Note on Difference Schemes

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A Pade' difference scheme which is up to sixth order accurate. Requires implicit inversion (tridiagonal) and special boundary operators. Sanjiva Lele developed the scheme and provided the basic equations. We start with scalar equations of the form

$$u_t + f(u)_x = 0 \tag{1}$$

The differencing scheme for $f(u)_x$ with $u_j^n = u(j\Delta x, n\Delta t)$ is

$$f'_{j-1} + \alpha f'_j + f'_{j+1} = \frac{B}{2\Delta x}\left(f_{j+1} - f_{j-1}\right) + \frac{C}{4\Delta x}\left(f_{j+2} - f_{j-2}\right) \tag{2}$$

with

$$B = \frac{4\alpha + 2}{3}, \quad C = \frac{4 - \alpha}{3}$$

This is a one parameter family of fourth order Pade' difference schemes. We can see this in a variety of ways. The modified equation error terms for the spatial difference can be found as

$$\frac{h^4}{10}\left(1 - \frac{\alpha}{3}\right) f_j^v + \frac{h^6}{1260}\left(8 - 5\alpha\right) f_j^{vii} + ...$$

which shows that for all α the scheme is at least 4^{th} order accurate and for $\alpha = 3.0$ the scheme is 6^{th} order. For $\alpha = 4$ we get a very standard 4^{th} order Pade' scheme. For other α we get a spectrum of schemes. We also see from the modified equation terms that the scheme is nondissipative (all error terms are odd derivatives).

Fourier analysis using $u_t + au = 0$ (i.e. the linear wave equation) where we use $u = e^{\lambda t} e^{ikx}$ results in

$$\lambda = -\frac{ai}{\Delta x}\frac{B \sin k\Delta x + \frac{C}{2}\sin 2k\Delta x}{\alpha + 2\cos k\Delta x}$$

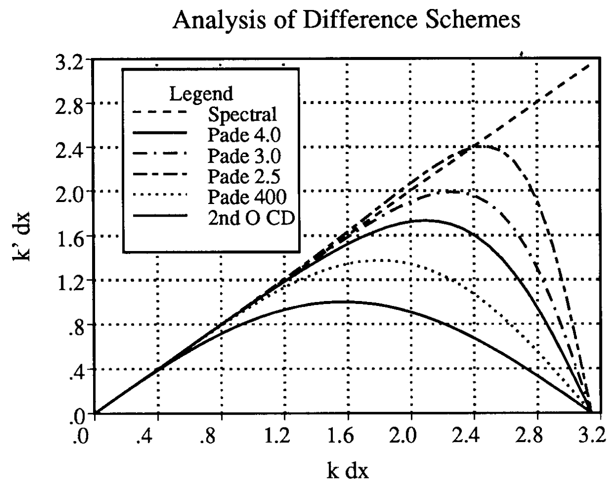

Figure 1: Modified Wave Number Analysis of High Order Pade' Scheme

For spectral differencing $u_x = ik$ and $\lambda_s = -aik$ and we can define the modified wave number (k') as $\lambda/(-ai)$ and plot k' vrs k to compare the wave number accuracy of various schemes. Figure 1 shows the relations for various α. At $\alpha = 4.0$ we have a common 4^{th} order Pade', for $\alpha = 3.0$ we have the 6^{th} order scheme and for $\alpha = 2.5$ the resulting scheme resolves more wave numbers then the others while maintaining formal 4^{th} order accuracy. For comparison purpose 2^{nd} order cental differencing is included where $k' = \sin k\Delta x/dx$. The important aspect of these results is the added resolution of wave number provided by the Pade' schemes. For example, with central differences we reach more than 2% error in the wave number after the first 18% of the full wave number resolution. The Pade' $\alpha = 4.0$ case 2% error point is at 41%, for $\alpha = 3.0$ the 2% error point occurs at 54% and for $\alpha = 2.65$ we get 69%. A range of $2.5 \leq \alpha \leq 3$ seems reasonable.